Question: (PreCalculus, or prior to introducing derivatives)

Algebraically determine the local maximum value of $f(x) = x^3 + 3x^2$.

Solution:

It may be helpful to think about this problem in terms of a vertical translation. If $f(x)$ is a polynomial with a local maximum at $x = k$, then there exists some vertical shift of c units that would cause it to be tangent to the x-axis at $x = k$.

If $f(x) + c$ is tangent to the x −axis at $x = k$, then it has a factor of $(x - k)$ with even multiplicity. Since the polynomial is cubic, the multiplicity must be 2. This gives us the equation:

$$x^3 + 3x^2 + c = (x - k)^2(x - a)$$

Where $x = a$ is some other root of the translated function. Expanding the right side of the equation and grouping like terms, we get:

$$x^3 + 3x^2 + c = x^3 - (2k + a)x^2 + (k^2 + 2ka)x - ak^2$$

Comparing the coefficients results in a system of equations:

$$3 = -(2k + a)$$

$$0 = k^2 + 2ka$$

$$c = -ak^2$$

The first two equations come from the x^2 and x coefficients, and they form a 2×2 system independent of c. Using the first equation to solve for a in terms of k, we get:

$$a = -3 - 2k$$

Substituting this equation into the second equation, we get:

$$0 = k^2 + 2(-3 - 2k)k$$

$$0 = -3k(k + 2)$$

$$k = 0, -2$$

Due to the independence of c in the first two equations of the system, this equation results in the location of both turning points in $f(x)$. If a cubic has two distinct turning points, one must be a local maximum and the other must be a local minimum. We only have to evaluate $f(x)$ at each of these values to determine which is the local maximum.

$f(0) = 0 \rightarrow$ local minimum

$f(-2) = 4 \rightarrow$ local maximum