Calculus Optimization Problems/Related Rates Problems Solutions

1) A farmer has 400 yards of fencing and wishes to fence three sides of a rectangular field (the fourth side is along an existing stone wall, and needs no additional fencing). Find the dimensions of the rectangular field of largest area that can be fenced.

$2x + y = 400 \Rightarrow y = 400 - 2x$

$A(x) = x(400 - 2x) = 400x - 2x^2$

$A'(x) = 400 - 4x \quad 400 - 4x = 0 \Rightarrow x = 100$

$A''(x) = -4$

By the 2nd derivative test, the dimensions would be 100 yd by 200 yd.

2) A metal box (without a top) is to be constructed from a square sheet of metal that is 20 cm on a side by cutting square pieces of the same size from the corners of the sheet and then folding up the sides. Find the dimensions of the box with the largest volume that can be constructed in this manner.

$V(x) = x(20 - 2x)(20 - 2x) = 400x - 80x^2 + 4x^3$

$V'(x) = 400 - 160x + 12x^2$

$400 - 160x + 12x^2 = 0 \Rightarrow 4(100 - 40x + 3x^2) = 0 \Rightarrow 4(3x - 10)(x - 10) \Rightarrow x = \frac{10}{3}, 10$

$V''(x) = -160 + 24x \quad V''\left(\frac{10}{3}\right) = -160 + 80 < 0 \quad V''(10) = -160 + 240 > 0$

By the 2nd derivative test, the dimensions would be $\frac{10}{3}$ cm by $\frac{40}{3}$ cm by $\frac{40}{3}$ cm

3) A rectangular field adjacent to a river is to be enclosed. Fencing along the river costs \$5 per meter, and the fencing for the other sides costs \$3 per meter. The area of the field is to be 1200 square meters. Find the dimensions of the field that is the least expensive to enclose.

Call the length of fence along the river *x*, and the length perpendicular to the river *y*.

$C(x) = 5x + 3(2y + x) \qquad xy = 1200 \Rightarrow y = \frac{1200}{x} \Rightarrow C(x) = 8x + \frac{7200}{x}$

$C'(x) = 8 - \frac{7200}{x^2} \quad 8 - \frac{7200}{x^2} = 0 \Rightarrow 8x^2 = 7200 \Rightarrow x^2 = 900 \Rightarrow x = 30$

$C''(x) = \frac{14400}{x^3} \quad C''(30) = \frac{14400}{30^3} > 0$

By the 2nd derivative test, a field that is 30 m along the river by 40 m perpendicular to the river would be least expensive.

4) A 4-meter length of stiff wire is cut in two pieces. One piece is bent into the shape of a square and the other into a rectangle whose length is 3 times its width. Let *x* be the length of the side of the square.
a) Find a formula *A*(*x*), the sum of the areas of the square and rectangle, in terms of the variable *x*.

The length of wire left for the rectangle is $4 - 4x$. In the rectangle, $l = 3w$. $4 - 4x = 2(3w) + 2w$, so $w = \frac{4-4x}{8} = \frac{1-x}{2} \Rightarrow l = \frac{3-3x}{2}$. $A(x) = x^2 + \left(\frac{3-3x}{2}\right)\left(\frac{1-x}{2}\right)$

b) For what values of x does $A(x)$ achieve its maximum; for which does it achieve its minimum. Justify your answer.

$$x^2+\left(\frac{3-3x}{2}\right)\left(\frac{1-x}{2}\right)=0 \Rightarrow x^2+\frac{3-6x+3x^2}{4}=0 \Rightarrow 7x^2-6x+3=0$$

5) A rectangular playing field is to have area 600 m^2. Fencing is required to enclose the field and to divide it into two equal halves.

a) Find a formula, $F(x)$, for the total length of fencing required, in terms of the length, x, of the fence dividing the field in half.

$$F(x)=3x+2\left(\frac{600}{x}\right)=3x+\frac{1200}{x}$$

b) Find the minimum amount of fencing needed to do this.

$$F'(x)=3-\frac{1200}{x^2} \quad 3-\frac{1200}{x^2}=0 \Rightarrow 3x^2=1200 \Rightarrow x=20$$

$$F''(x)=\frac{2400}{x^3} \quad F''(20)=\frac{2400}{20^3}>0$$

By the 2nd derivative test, The minimum amount of fencing needed is 120 m

c) What are the outer dimensions of the field that has the least fencing? 20 m by 30 m

6) A rectangle has its base on the x-axis and its upper vertices on the parabola $y = 27 - x^2$. Find the maximum possible area of the rectangle.

$$A(x)=2x\left(27-x^2\right)=54x-2x^3$$

$$A'(x)=54-6x^2 \quad 54-6x^2=0 \Rightarrow x^2=9 \Rightarrow x=3$$

$$A''(x)=-12x \quad A''(3)=-36<0$$

By the 2nd derivative test, the maximum area would be 6(18) = 108 sq units.

7) A rectangular container with open top is required to have a volume of 16 cubic meters. Also, one side of the rectangular base is required to be 4 meters long. If material for the base costs \$8 per square meter, and material for the sides costs \$2 per square meter, find the dimensions of the container so that the cost of material to make it will be a minimum.

$$V=4wh=16 \Rightarrow h=\frac{4}{w}$$

$$C=8(4w)+2(2wh)+2(2(4h))=32w+16+\frac{64}{w}$$

$$C'=32-\frac{64}{w^2} \quad 32-\frac{64}{w^2}=0 \Rightarrow w=\sqrt{2}$$

$$C''=\frac{128}{w^3} \quad \frac{128}{\sqrt{2}^3}>0$$

By the 2nd derivative test, the dimensions of the container that minimizes the cost are 4 m by $\sqrt{2}$ m (base) by $\frac{4}{\sqrt{2}}$ m (height)

8) A rectangular box with open top is to be constructed from a rectangular piece of cardboard 80 cm by 30 cm, by cutting out equal squares from each corner of the sheet of cardboard and folding up the resulting flaps. Find the dimensions of the box of maximum volume made by these conditions.

$$V = x(80-2x)(30-2x) = 2400x - 220x^2 + 4x^3$$

$$V' = 2400 - 440x + 12x^2 = 4(3x^2 - 110x + 600)$$

$$4(3x^2 - 110x + 600) = 0 \Rightarrow 4(3x-20)(x-30) = 0 \Rightarrow x = \frac{20}{3}, 30$$

$$V''(x) = -440 + 24x \quad V''\left(\frac{20}{3}\right) = -440 + 160 < 0$$

By the 2nd derivative test, the dimension of the box of maximum volume are $\frac{20}{3}$ cm by $\frac{200}{3}$ cm by $\frac{50}{3}$ cm

9) Find the points on the parabola $2x + y^2 = 0$ closest to the point (-3,0).

10) A power line is needed to connect a power station on the shore of a river to an island 4 miles downstream and 1 mile offshore. Find the minimum cost for such a line given that is costs $50,000 per mile to lay wire under the water and $30,000 per mile to lay wire underground.